

January 2009 - Solutions

1) Solve the equation $x^{\log_{27} 8} + 8^{\log_{27} x} = 32$.

Solution:

$$\left(27^{\log_{27} x}\right)^{\log_{27} 8} + 8^{\log_{27} x} = 32$$

$$27^{(\log_{27} x)(\log_{27} 8)} + 8^{\log_{27} x} = 32$$

$$\left(27^{\log_{27} 8}\right)^{\log_{27} x} + 8^{\log_{27} x} = 32$$

$$8^{\log_{27} x} + 8^{\log_{27} x} = 32$$

$$2 \cdot 8^{\log_{27} x} = 32$$

$$8^{\log_{27} x} = 16$$

$$\log_{27} x = \frac{4}{3}$$

$$x = 27^{\frac{4}{3}} = 81$$

The solution set is $\{ 81 \}$

Check: $81^{\log_{27} 8} + 8^{\log_{27} 81} = 32$

$$\left(27^{\frac{4}{3}}\right)^{\log_{27} 8} + 8^{\frac{4}{3}} = 32$$

$$27^{\frac{4}{3} \log_{27} 8} + 8^{\frac{4}{3}} = 32$$

$$\left(27^{\log_{27} 8}\right)^{\frac{4}{3}} + 8^{\frac{4}{3}} = 32$$

$$8^{\frac{4}{3}} + 8^{\frac{4}{3}} = 32$$

$$16 + 16 = 32$$

2) Find as many sets of three positive integers $\{a, b, c\}$ as you can such that $a^2 + b^2 = c^2$, and $a, b,$ and c are relatively prime. ($a, b,$ and c are relatively prime if they have no common positive factors except 1).

EXAMPLES: $\{3, 4, 5\}$ and $\{5, 12, 13\}$ would count. However, $\{6, 8, 10\}$ would not count, since 6, 8, and 10 each have 2 as a factor.

Solution: Following are all such sets with $c < 200$. There are others with $c \geq 200$.

$\{3, 4, 5\}$	$\{5, 12, 13\}$	$\{7, 24, 25\}$	$\{8, 15, 17\}$	$\{9, 40, 41\}$
$\{11, 60, 61\}$	$\{12, 35, 37\}$	$\{13, 84, 85\}$	$\{15, 112, 113\}$	$\{16, 63, 65\}$
$\{17, 144, 145\}$	$\{19, 180, 181\}$	$\{20, 21, 29\}$	$\{20, 99, 101\}$	$\{24, 143, 145\}$
$\{28, 45, 53\}$	$\{33, 56, 65\}$	$\{36, 77, 85\}$	$\{39, 80, 89\}$	$\{44, 117, 125\}$
$\{48, 55, 73\}$	$\{51, 140, 149\}$	$\{60, 91, 109\}$	$\{65, 72, 97\}$	$\{88, 105, 137\}$

3) Find positive integers $a, b, c,$ and $d,$ such that $a^3 + b^3 + c^3 = d^3$.

Solution: Here are some solutions. There are more.

$3^3 + 4^3 + 5^3 = 6^3$	$1^3 + 6^3 + 8^3 = 9^3$	$2^3 + 17^3 + 40^3 = 41^3$
$3^3 + 10^3 + 18^3 = 19^3$	$3^3 + 36^3 + 37^3 = 46^3$	$4^3 + 17^3 + 22^3 = 25^3$
$6^3 + 32^3 + 33^3 = 41^3$	$7^3 + 14^3 + 17^3 = 20^3$	$11^3 + 15^3 + 27^3 = 29^3$
$16^3 + 23^3 + 41^3 = 44^3$	$18^3 + 19^3 + 21^3 = 28^3$	$27^3 + 30^3 + 37^3 = 46^3$