February 2009 - Solutions

1) When I am as old as my father is now, I will be six times as old as my son is now. By then, my son will be two years older than I am now. The sum of my father's age and my age is 94 years. How much older am I than my son?

Solution:

Let m = my present age. Let f = my father's present age. Let s = my son's present age.

$f - m$ = the difference of my father's age and my age

$s + (f - m)$ = my son's age when I am as old as my father is now

We have the equation $s + f - m = m + 2$, which can be rewritten as $s + f - 2m = 2$.

We must solve the system of linear equations:

(1) $f = 6s$
(2) $s + f - 2m = 2$
(3) $f + m = 94$

Substitute $6s$ for f in (2) and (3):

$s + 6s - 2m = 2$

(4) $7s - 2m = 2$
(5) $6s + m = 94$

$m = 94 - 6s$. Substituting this expression for m in (4):

$7s - 2(94 - 6s) = 2$

$7s - 188 + 12s = 2$

$19s = 190$

$s = 10$

$m = 94 - 6 \cdot 10 = 34$

$f = 6 \cdot 10 = 60$

When I am as old as my father is now, 60, I will be six times as old as my son is now, 10. Since I am 34, this will be in 26 years. By then, my son will be 36, two years older than I am now. The sum of my father's age and my age is 60 + 34 = 94 years.

I am 34 – 10 = 24 years older than my son.

2) Solve the system of equations.

(1) $4x = 3y$
(2) $x^2 + y^2 + z^2 = 1$
(3) $x^2 + y^2 = 25(1 - z)^2$

Solution:

We seek real numbers x, y, and z such that (x, y, z) is a solution of (1), (2), and (3).

$25(1 - z)^2 + z^2 = 1$

$25(1 - z)^2 + z^2 - 1 = 0$

$25(z - 1)^2 + z^2 - 1 = 0$

$25(z - 1)^2 + (z + 1)(z - 1) = 0$

$(z - 1)[25(z - 1) + (z + 1)] = 0$

$(z - 1)(26z - 24) = 0$

$2(z - 1)(13z - 12) = 0$

Either i) $z = 1$ or ii) $z = \frac{12}{13}$

i) $z = 1$

$x^2 + y^2 + 1^2 = 1$

$x^2 + y^2 = 0$

$x = y = 0$

We verify that (0, 0, 1) is a solution:

(1) $4 \cdot 0 = 3 \cdot 0$ $\quad 0 = 0$
(2) $0^2 + 0^2 + 1^2 = 1$ $\quad 1 = 1$
(3) $0^2 + 0^2 = 25(1 - 1)^2$ $\quad 0 = 25 \cdot 0$ $\quad 0 = 0$

ii) $z = \frac{12}{13}$

Multiply (2) by 9:

$9x^2 + 9y^2 + 9z^2 = 9$

$9x^2 + 16x^2 + 9\left(\frac{12}{13}\right)^2 = 9$

$$25x^2 = 9\left[1 - \left(\frac{12}{13}\right)^2\right]$$

$$25x^2 = 9\left[1 - \frac{144}{169}\right]$$

$$25x^2 = 9 \cdot \frac{169-144}{169}$$

$$25x^2 = 9 \cdot \frac{25}{169}$$

$$x^2 = 9 \cdot \frac{1}{169} = \frac{9}{169}$$

$$x = \pm \frac{3}{13}$$

From (1):

$$4 \cdot \left(\pm \frac{3}{13}\right) = \pm \frac{12}{13} = 3y$$

$$y = \pm \frac{4}{13}$$

We verify that $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ is a solution:

(1) $4 \cdot \frac{3}{13} = 3 \cdot \frac{4}{13}$

$$\frac{12}{13} = \frac{12}{13}$$

(2) $\left(\frac{3}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$

$$\frac{9}{169} + \frac{16}{169} + \frac{144}{169} = 1$$

$$\frac{169}{169} = 1$$

$1 = 1$

(3) $\left(\frac{3}{13}\right)^2 + \left(\frac{4}{13}\right)^2 = 25\left(1 - \frac{12}{13}\right)^2$

$$\frac{9}{169} + \frac{16}{169} = 25\left(1 - \frac{12}{13}\right)^2$$

$$\frac{25}{169} = 25\left(\frac{1}{13}\right)^2$$

$$\frac{25}{169} = 25 \cdot \frac{1}{169}$$

$$\frac{25}{169} = \frac{25}{169}$$

We verify that $\left(-\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)$ is a solution:

(1) $4\left(-\frac{3}{13}\right) = 3\left(-\frac{4}{13}\right)$

$$-\frac{12}{13} = -\frac{12}{13}$$

(2) $\left(-\frac{3}{13}\right)^2 + \left(-\frac{4}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$

$$\frac{9}{169} + \frac{16}{169} + \frac{144}{169} = 1$$

$$\frac{169}{169} = 1$$

$1 = 1$

(3) $\left(-\frac{3}{13}\right)^2 + \left(-\frac{4}{13}\right)^2 = 25\left(1 - \frac{12}{13}\right)^2$

$$\frac{9}{169} + \frac{16}{169} = 25\left(1 - \frac{12}{13}\right)^2$$

$$\frac{25}{169} = 25\left(\frac{1}{13}\right)^2$$

$$\frac{25}{169} = 25 \cdot \frac{1}{169}$$

$$\frac{25}{169} = \frac{25}{169}$$

The solution set is $\left\{(0, 0, 1),\ \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right), \left(-\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)\right\}$

3) Let u and v be real numbers. Solve the system of equations for x, y, and z in terms of u and v.

(1) $vx = uy$
(2) $x^2 + y^2 + z^2 = 1$
(3) $x^2 + y^2 = (u^2 + v^2)(1 - z)^2$

Solution:

We seek real numbers x, y, and z such that (x, y, z) is a solution of (1), (2), and (3).

$(u^2 + v^2)(1 - z)^2 + z^2 = 1$

$(u^2 + v^2)(1 - z)^2 + z^2 - 1 = 0$

$(u^2 + v^2)(z - 1)^2 + z^2 - 1 = 0$

$(u^2 + v^2)(z - 1)^2 + (z + 1)(z - 1) = 0$

$(z - 1)[(u^2 + v^2)(z - 1) + (z + 1)] = 0$

Either i) $z - 1 = 0$ or ii) $(u^2 + v^2)(z - 1) + (z + 1) = 0$.

i) $z - 1 = 0$

$z = 1$

$x^2 + y^2 + 1^2 = 1$

$x^2 + y^2 = 0$

$x = y = 0$

We verify that (0, 0, 1) is a solution:

(1) $v \cdot 0 = u \cdot 0 \qquad 0 = 0$

(2) $0^2 + 0^2 + 1^2 = 1 \qquad 1 = 1$

(3) $0^2 + 0^2 = (u^2 + v^2)(1 - 1)^2 \qquad 0 = (u^2 + v^2) \cdot 0 \qquad 0 = 0$

ii) $(u^2 + v^2)(z - 1) + (z + 1) = 0$

$(u^2 + v^2)z - (u^2 + v^2) + z + 1 = 0$

$(u^2 + v^2 + 1)z = u^2 + v^2 - 1$

$$z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

Let $u \neq 0$ and multiply (2) by u^2:

$$u^2 x^2 + u^2 y^2 + u^2 z^2 = u^2$$

$$u^2 x^2 + v^2 x^2 + u^2 z^2 = u^2$$

$$(u^2 + v^2)x^2 + u^2 \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 = u^2$$

$$(u^2 + v^2)x^2 = u^2 - u^2 \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2$$

$$(u^2 + v^2)x^2 = u^2 \left[1 - \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2\right]$$

$$(u^2 + v^2)x^2 = u^2 \left[\frac{(u^2+v^2+1)^2}{(u^2+v^2+1)^2} - \frac{(u^2+v^2-1)^2}{(u^2+v^2+1)^2}\right]$$

$$(u^2 + v^2)x^2 = u^2 \left[\frac{[(u^2+v^2)^2+2(u^2+v^2)+1]-[(u^2+v^2)^2-2(u^2+v^2)+1]}{(u^2+v^2+1)^2}\right]$$

$$(u^2 + v^2)x^2 = u^2 \left[\frac{4(u^2+v^2)}{(u^2+v^2+1)^2}\right]$$

Since $u \neq 0$, $u^2 + v^2 \neq 0$, and it follows that:

$$x^2 = u^2 \left[\frac{4}{(u^2+v^2+1)^2}\right] = \frac{4u^2}{(u^2+v^2+1)^2}$$

$$x = \pm \frac{2u}{u^2+v^2+1}$$

From (1):

$$\pm \frac{2uv}{u^2+v^2+1} = uy$$

Since $u \neq 0$:

$$y = \pm \frac{2v}{u^2+v^2+1}$$

We verify that $\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ is a solution:

(1) $v\left[\frac{2u}{u^2+v^2+1}\right] = u\left[\frac{2v}{u^2+v^2+1}\right]$

$$\frac{2uv}{u^2+v^2+1} = \frac{2uv}{u^2+v^2+1}$$

(2) $\left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 = 1$

$$\frac{4u^2}{(u^2+v^2+1)^2} + \frac{4v^2}{(u^2+v^2+1)^2} + \frac{(u^2+v^2)^2-2(u^2+v^2)+1}{(u^2+v^2+1)^2} = 1$$

$$\frac{4u^2+4v^2+(u^2+v^2)^2-2u^2-2v^2+1}{(u^2+v^2+1)^2} = 1$$

$$\frac{(u^2+v^2)^2+2u^2+2v^2+1}{(u^2+v^2+1)^2} = 1$$

$$\frac{(u^2+v^2)^2+2(u^2+v^2)+1}{(u^2+v^2+1)^2} = 1$$

$$\frac{(u^2+v^2+1)^2}{(u^2+v^2+1)^2} = 1$$

$$1 = 1$$

(3) $\left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 = (u^2+v^2)\left(1-\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2$

$$\frac{4u^2+4v^2}{(u^2+v^2+1)^2} = (u^2+v^2)\left(\frac{(u^2+v^2+1)-(u^2+v^2-1)}{u^2+v^2+1}\right)^2$$

$$\frac{4u^2+4v^2}{(u^2+v^2+1)^2} = (u^2+v^2)\left(\frac{2}{u^2+v^2+1}\right)^2$$

$$\frac{4u^2+4v^2}{(u^2+v^2+1)^2} = (u^2+v^2)\frac{4}{(u^2+v^2+1)^2}$$

$$\frac{4u^2+4v^2}{(u^2+v^2+1)^2} = \frac{4u^2+4v^2}{(u^2+v^2+1)^2}$$

We verify that $\left(-\frac{2u}{u^2+v^2+1}, -\frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ is a solution:

(1) $v\left[-\frac{2u}{u^2+v^2+1}\right] = u\left[-\frac{2v}{u^2+v^2+1}\right]$

$$-\frac{2uv}{u^2+v^2+1} = -\frac{2uv}{u^2+v^2+1}$$

(2) $\left(-\frac{2u}{u^2+v^2+1}\right)^2 + \left(-\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 = 1$

$$\left(-\frac{2u}{u^2+v^2+1}\right)^2 + \left(-\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 = \left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2$$

From above, $\left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 = 1$

1 = 1

(3) $\left(-\frac{2u}{u^2+v^2+1}\right)^2 + \left(-\frac{2v}{u^2+v^2+1}\right)^2 = (u^2+v^2)\left(1-\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2$

$$\left(-\frac{2u}{u^2+v^2+1}\right)^2 + \left(-\frac{2v}{u^2+v^2+1}\right)^2 = \left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2$$

From above, $\left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 = (u^2+v^2)\left(1-\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2$

$$(u^2+v^2)\left(1-\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 = (u^2+v^2)\left(1-\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2$$

Notice , in the verifications for $\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ and $\left(-\frac{2u}{u^2+v^2+1}, -\frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$, that u may be 0.

We have that (0, 0, 1), $\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ and $\left(-\frac{2u}{u^2+v^2+1}, -\frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ are solutions, where u and v can be any real number.

We must verify that there are no additional solutions when u = 0.

Let v ≠ 0. Then, by symmetry, we again obtain the solutions $\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ and $\left(-\frac{2u}{u^2+v^2+1}, -\frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$.

Let u = v = 0. Then from (3):

$$x^2 + y^2 = (u^2 + v^2)(1 - z)^2 = 0(1 - z)^2 = 0$$

x = y = 0

From (2):

$$x^2 + y^2 + z^2 = 0 + z^2 = 1$$

$$z^2 = 1$$

$$z = \pm 1$$

We have the solutions (0, 0, 1) and (0, 0, –1). However, neither solution is new, since if u = v = 0, then

$$\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) = \left(-\frac{2u}{u^2+v^2+1}, -\frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) = (0, 0, -1).$$

The solution set is $\left\{(0,0,1),\ \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right),\ \left(-\frac{2u}{u^2+v^2+1}, -\frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)\right\}$.