April 2009 - Solutions

1) A positive integer less than 1000 is chosen at random. What is the probability it is a multiple of 7, but a multiple of neither 3 nor 14?

Solution:

Let x be an integer. Let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x.
Let S = the set of positive integers less than 1000.
Let n_7 be the number of integers in S which are divisible by 7.
Let n be the number of integers in S which are divisible by 7 and are divisible by either 3 or 14.

The answer to our problem is $\frac{n_7 - \mathrm{n}}{999}$. $n_7 = \lfloor 999/7 \rfloor = 142$.

We now need only to compute n, and we will do that next:

Let n_{21} be the number of integers in S which are divisible by 7 and 3.
Let n_{14} be the number of integers in S which are divisible by 7 and 14.

An integer is divisible by 7 and divisible by 3 if and only if it is divisible by 21, and an integer is divisible by 7 and divisible by 14 if and only if it is divisible by 14. Therefore:

$n_{21} = \lfloor 999/21 \rfloor = 47$.
$n_{14} = \lfloor 999/14 \rfloor = 71$.
n = the number of integers in S which are divisible by 21 or 14.

Let n_{42} be the number of integers in S which are divisible by 21 and 14.

An integer is divisible by 21 and divisible by 14 if and only if it is divisible by 42.

$n_{42} = \lfloor 999/42 \rfloor = 23$.

n = $n_{21} + n_{14} - n_{42} = 47 + 71 - 23 = 95$.

The answer to our problem is $\frac{142-95}{999} = \boxed{\frac{47}{999}} \approx .04705$.

2) The sum of the absolute values of all solutions of the equation $|x^3 + 4x^2 - 12x - 44| = x^2 + 4x + 4$ can be written in the form a + b$\sqrt{c}$, c a prime. Find a + b + c.

Solution:

(0) $|x^3 + 4x^2 - 12x - 44| = x^2 + 4x + 4$

(i) $x^3 + 4x^2 - 12x - 44 = x^2 + 4x + 4$ and $x^3 + 4x^2 - 12x - 44 \geq 0$

(ii) $x^3 + 4x^2 - 12x - 44 = -x^2 - 4x - 4$ and $x^3 + 4x^2 - 12x - 44 \leq 0$

(1) $x^3 + 4x^2 - 12x - 44 = x^2 + 4x + 4$

(2) $x^3 + 4x^2 - 12x - 44 = -x^2 - 4x - 4$

(0) is true if and only (i) is true or (ii) is true.
Since $x^2 + 4x + 4 = (x + 2)^2 \geq 0$ for all real numbers x, (i) is true if and only if (1) is true.
Since $-x^2 - 4x - 4 = -(x + 2)^2 \leq 0$ for all real numbers x, (ii) is true if and only if (2) is true.

Therefore, a real number is a solution of (0) if and only if it is a solution of (1) or it is a solution of (2).

First we solve (1):

$x^3 + 4x^2 - 12x - 44 = x^2 + 4x + 4$
$x^3 + 3x^2 - 16x - 48 = 0$
$(x + 3)(x^2 - 4) = 0$
x = –3, –2, 2

Next we solve (2):

$x^3 + 4x^2 - 12x - 44 = -x^2 - 4x - 4$
$x^3 + 5x^2 - 8x - 40 = 0$
$(x + 5)(x^2 - 8) = 0$
x = –5, $-2\sqrt{2}$, $2\sqrt{2}$

The sum of the absolute values of all solutions is $3 + 2 + 2 + 5 + 2\sqrt{2} + 2\sqrt{2} = 12 + 4\sqrt{2}$.

a = 12. b = 4. c = 2.

a + b + c = 18